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# MASS TRANSFER THROUGH LAMINAR BOUNDARY LAYERS—6. METHODS OF EVALUATING THE WALL GRADIENT ($b_0'/B$) FOR SIMILAR SOLUTIONS; SOME NEW VALUES FOR ZERO MAIN-STREAM PRESSURE GRADIENT

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**Abstract**—The wall gradient ($b_0'/B$) of the $b$-boundary layer is evaluated from solutions to the velocity equation by integration. By dividing the range of integration into two parts (a) that over the velocity boundary layer and (b) that for main-stream flow, a formula is derived from which this wall gradient can be evaluated accurately for any value of the Prandtl/Schmidt number. Methods are then given for doing this either with a desk calculator or on a computer. These methods are not convenient however, when the Prandtl/Schmidt number has a high value and there is a high rate of inward mass flow; an asymptotic formula is therefore derived for this case. A table of values of the wall gradient, most of which are new, is then given for the case of zero main-stream pressure gradient. A discussion follows of the asymptotic behaviour of functions of the $b$-boundary layer when various parameters take extreme values and some of these functions are plotted.

**Résumé**—Le gradient à la paroi ($b_0'/B$) de la couche limite est évalué, par intégration, à partir des solutions de l'équation du mouvement. En divisant l'intervalle d'intégration en deux parties: (a) celui qui couvre la couche limite dynamique et (b) celui relatif à l'écoulement principal, on établit une formule à partir de laquelle on peut calculer avec précision le gradient à la paroi, pour une valeur quelconque du nombre de Prandtl/Schmidt. Des méthodes sont ensuite données pour effecteur ce calcul soit à l'aide d'une machine à calculer de bureau soit avec une calculatrice. Toutefois, ces méthodes ne conviennent pas quand le nombre de Prandtl/Schmidt est élevé et lorsqu'il existe un transport de masse important dans la couche limite; pour ce cas, une formule asymptotique a été établie. Une table des valeurs du gradient à la paroi, dont la plupart sont nouvelles, est ensuite donnée quand le gradient de pression est nul dans l'écoulement principal. Suit une discussion sur le comportement asymptotique des fonctions dans la couche limite $b$ lorsque les différents paramètres prennent des valeurs extremes; quelques unes de ces fonctions ont été tracées.

**Zusammenfassung**—Durch Integration von Lösungen der Geschwindigkeitsgleichung wurde der Wandgradient ($b_0'/B$) der $b$-Grenzschicht berechnet. Teilt man den Integrationsbereich (a) in den Bereich der Geschwindigkeitsgrenzschicht und (b) in den Bereich der Hauptströmung, so erhält man eine für beliebige Werte der Prandtl/Schmidtzahl gültige Formel fèr die exakten Werte des Wandgradienten. Diese Berechnung ist für Hand- und Elektronenrechner angegeben. Für grosse Werte der Prandtl/Schmidtzahl und einen grossen, nach innen durch die Granzfläche tretenden Massenstrom ist diese Berechnungsmethode unbequem, deshalb wurde dafür eine asymptotische Formel abgeleitet. Eine Tabelle vorwiegend neuer Werte des Wandgradienten ist für einen Druckgradienten Null der Hauptströmung angegeben. Für Extremwerter einiger Parameter zeigen die Funktionen der $b$-Grenzschicht ein asymptotisches Verhalten. Einige dieser Funktionen sind als Diagramme mitgeteilt.

**Аннотация**—Градиент ($b_0'/B$) пограничного слоя $b$ на стенке вычисляется из решений уравнения скорости путём его интегрирования. Разделив интегралы на две части: (а) по пограничному слою и (б) для основного потока, получим формулу, из которой можно точно вычислить градиент на стенке для любых чисел Прандтля/Шмндта. Также в статье приводится методика вычисления с помощью счётни-вычисли-тельных машип. Однако, эти методы непригодны в случае больших чисел Прандтля/Шмидта или большой скорости течения; в этом случае выводится асимптотическая формула. В конце статьи приведена таблица величин градиента на стенке, большая часть из которых даётся впервые, для случая нулевого градиента давления в основном потоке. Рассматривается асимптотическое поведение функций иограничного слоя $b$, когда различные параметры принимают крайние значения, а некоторые функции даны графически.

## NOTATION

Most of the quantities given below are dimensionless; where they are not the dimensions are given in brackets.

- $a_m$, coefficients occurring in equation (23) and defined in equations (26) to (33);
- $A_q$, coefficients occurring in equation (20) defined in equation (21);
- $b$, conserved fluid property in dimensionless form (see Paper 3 for a discussion of its form and meaning);
- $b_0'$, gradient of $b$ in the fluid at the interface; see equation (10);
- $B$, value of $b$ in the main-stream; it is the driving force for mass transfer (discussed fully in Paper 3);
- $C$, constant occurring in equation (1);
- $c$, abbreviation for $(f_0''/f_0^3)$ which occurs in equations (26) to (34);
- $d$, dimensionless distance occurring in equation (15); it is the value of the "similar" co-ordinate $\eta$ at which the flow can be regarded as inviscid;
- $e$, abbreviation for $(\beta/f_0^4)$ which occurs in equations (26) to (34);
- $f$, dimensionless stream function in "similar" co-ordinates defined in equation (4);
- $f_0$, value of $f$ at the interface; see equation (7);
- $f_0'', f_0^{(n)}$, values of the derivatives of $f$ at the interface;
- $F_2$, function which measures the rate of growth with distance $x$ of the momentum boundary layer thickness $\delta_2$; see equation (57) and Appendix;
- $K$, diffusion constant; the thermal diffusivity of the fluid for heat transfer, the diffusion coefficient of a mass component in the fluid mixture for mass transfer ($ft^2/h$);
- $n$, constant occurring in equation (1);
- $Nu$, Nusselt number in terms of the length $x$;
- $p$, number occurring in equation (22) specifying terms in the expanded form of equation (20);
- $q$, number occurring in equation (21) specifying terms in equation (20);
- $Re$, local Reynolds number $(=u_G x/\nu)$;
- $u$, velocity component parallel to the interface (ft/h);
- $u_G$, value of $u$ in the main-stream (ft/h);
- $v$, velocity component normal to the interface (ft/h);
- $v_0$, value of $v$ at the interface (ft/h);
- $W$, curvature parameter in terms of the convection thickness $\Delta_2$; equation (46);
- $x$, distance parallel to the interface measured from the start of the boundary layer (ft);
- $X$, curvature parameter in terms of the conduction thickness $\Delta_4$; equation (44);
- $y$, distance measured perpendicular to the interface (ft);
- $Y$, function which is a measure of the rate of growth with distance $x$ of the conduction thickness $\Delta_4$; equation (45);
- $Z$, function which is a measure of the rate of growth with distance $x$ of the convection thickness $\Delta_2$; equation (47).

Greek symbols

- $\beta$, parameter occurring in the velocity equation; it is a measure of the main-stream pressure gradient; equation (2);
- $\gamma$, fluid property called the "exchange coefficient" given by $K\rho$, (lb/ft h);
- $\delta_1^*$, displacement boundary layer thickness in "similar" co-ordinates; defined by equation (13), (ft);
- $\delta_2$, momentum boundary layer thickness,

$$= \int_0^\infty \frac{u}{u_G}\left(1 - \frac{u}{u_G}\right) dy, \quad \text{(ft)};$$

- $\delta_4$, shear boundary layer thickness, $= u_G/(\partial u/\partial y)_0$, (ft);

$\Delta_2$, convection boundary layer thickness, $= \int_0^\infty \frac{u}{u_G}\left(1 - \frac{b}{B}\right) dy$, (ft);

$\Delta_4$, conduction boundary layer thickness, $= B/(\partial b/\partial y)_0$, (ft);

$\eta$, "similar" dimensionless length co-ordinate which gives the distance of a point from the interface, equation (3);

$\mu$, dynamic viscosity of fluid (lb/ft h);

$\nu$, kinematic viscosity of fluid $(=\mu/\rho)$, ($ft^2$/h);

$\rho$, density of fluid (lb/$ft^3$);

$\sigma$, Prandtl or Schmidt number $(=\nu/K)$;

$\varphi$, integration variable used in section 3.4, defined in equation (18);

$\psi$, stream function ($ft^2$/h).

Subscripts

G, denotes fluid state in the main-stream;

$m$, denotes terms in equation (23);

0, denotes fluid state adjacent to the interface;

$q$, denotes terms in equation (20).

## 1. INTRODUCTION

### 1.1. *General remarks*

In earlier papers of the present series it was shown that when the fluid properties are uniform the prediction of mass transfer rates through laminar boundary layers can be reduced to the solution of two simultaneous partial differential equations. The velocity equation governs the distribution in the boundary layer of momentum, shear and other purely mechanical quantities; this was considered in Paper 1, Spalding [1] and Paper 2, Spalding and Evans [2]. The $b$-equation governs the distribution of other conserved fluid properties; this was discussed in Paper 3, Spalding and Evans [3].

It was also shown that for "similar" flows these equations reduce to ordinary differential equations which can be solved exactly. These "similar" solutions are important not only in their application to strictly "similar" flows but also serve as a basis for more general, if approximate, methods applicable to problems involving non-similar flows.

An important quantity occurring in the $b$-equation is the "wall gradient" $(b'_0/B)$, where $b'_0$ is the gradient of the conserved fluid property $b$ with respect to the "similar" dimensionless distance $\eta$, the suffix denoting the value at $\eta = 0$, and $B$ is the value of $b$ in the main-stream. In principle this wall gradient can be evaluated by integration once the distribution of the stream function is known.

In Paper 3 it was shown that comparatively few exact values of the wall gradient could be found in the literature. Most of them related to the case of zero main-stream pressure gradient and were confined to a fairly narrow range near unity in the Prandtl/Schmidt number $\sigma$. Paper 3 also contained tables from which approximate values of $(b'_0/B)$ could be obtained for wide ranges in the main-stream pressure gradient and Prandtl/Schmidt number.

Paper 3a, Evans [4], was concerned with the case of zero mass transfer. Series were given from which the wall gradient could be evaluated accurately for wide ranges in the main-stream pressure gradient and for any value greater than 0·5 of the Prandtl/Schmidt number $\sigma$.

### 1.2. *Outline of the present paper*

The main aims of the present paper are to give methods of evaluating the wall gradient $(b'_0/B)$ accurately and to present a table of values for the case of zero main-stream pressure gradient. This table covers wide ranges in the fluid Prandtl/Schmidt number as well as the mass transfer rate. Formulae and methods of calculation are given in a general form which hold even when the main-stream pressure gradient is not zero, but results for non-zero values of this parameter are to be given in later papers.

After a brief statement of the mathematical problem which has to be solved, an expression is derived for the reciprocal of the wall gradient which greatly simplifies the problem of numerical evaluation. This can be used for any values of the parameters specifying the Prandtl/Schmidt number, main-stream pressure gradient or mass transfer rate.

Values of the wall gradient have been obtained in three ways. When solutions to the velocity equation are known, calculations with a desk calculator can give high accuracy; the method used is described in section 3.2. Most of the results were, however, obtained on a computer the method for which is outlined in section 3.3. But, when the Prandtl/Schmidt number has a high value and the suction rate is high, neither of these methods is suitable so an asymptotic formula which gives high accuracy under these conditions is derived in section 3.4.

The results obtained for zero main-stream pressure gradient are discussed in section 4 and formulae are given in section 5 from which other functions of the $b$-boundary layer may be calculated. Since these functions are readily evaluated from the wall gradient and the known values of the various parameters, they are not tabulated in the present paper.

In section 6 an examination is made of the asymptotic behaviour of functions in the $b$-boundary layer for extreme values of the parameters and in section 7 some of these functions are plotted and discussed.

## 2. STATEMENT OF THE MATHEMATICAL PROBLEM

The forms which the two-dimensional, laminar boundary layer equations take for "similar" flows have been fully discussed in earlier papers in the present series. For purposes of reference and in order to explain the terminology used, the equations are stated briefly in the present section.

### 2.1. *The "similar" velocity equation*

In Paper 1, Spalding [1], it was shown that for "similar" solutions to the two-dimensional, laminar boundary layer equations with constant fluid properties, the main-stream velocity $u_G$ obeys the equation:

$$\frac{du_G}{dx} = C\,u_G^n \tag{1}$$

where $C$ and $n$ are constants and $x$ is the distance measured parallel to the wall.

A parameter $\beta$ is obtained from the constant $n$ by the relationship:

$$\beta = \frac{1}{\left(1 - \frac{n}{2}\right)} \tag{2}$$

and a dimensionless distance co-ordinate $\eta$ is defined by:

$$\eta = y\left(\frac{1}{\nu\beta}\cdot\frac{du_G}{dx}\right)^{1/2} \tag{3}$$

where $y$ is the distance perpendicular to the wall and $\nu$ is the kinematic viscosity. If now a dimensionless stream function $f$ is related to the stream function $\psi$ by:

$$f = \frac{\psi}{u_G}\left(\frac{1}{\nu\beta}\frac{du_G}{dx}\right)^{1/2} \tag{4}$$

velocities in the boundary layer are governed by the ordinary differential equation:

$$f''' + ff'' + \beta(1 - f'^2) = 0 \tag{5}$$

with the boundary conditions:

$$\left.\begin{array}{ll} \eta = 0, & f = f_0,\quad f' = 0 \\ \eta \to \infty, & f' \to 1. \end{array}\right\} \tag{6}$$

In equations (5) and (6) the primes denote differentiation with respect to the independent variable $\eta$ and the quantity $f_0$ in equation (6), which is a constant for "similar" flows, is related to the velocity $v_0$ with which mass flows through the wall by:

$$f_0 = -v_0\left(\frac{\nu}{\beta}\frac{du_G}{dx}\right)^{-1/2}. \tag{7}$$

### 2.2. *The "similar" b-equation*

This equation is a generalization of the energy equation familiar in the study of heat transfer. It has been fully discussed by Spalding [5] and "similar" solutions to it were considered in Paper 3. In the latter paper it was shown that for a certain restricted class of "similar" solutions a conserved fluid property, represented in suitable dimensionless form by the quantity $b$, satisfies the ordinary differential equation:

$$b'' + \sigma f b' = 0 \tag{8}$$

with boundary conditions:

$$\left.\begin{array}{ll} \eta = 0, & b = 0 \\ \eta \to \infty, & b \to B \end{array}\right\}. \tag{9}$$

In addition to these boundary conditions the relationship:

$$b_0' = -\sigma f_0, \tag{10}$$

which relates the $b$-profile to the $f$-profile, is also satisfied.

In these equations the primes again denote differentiation with respect to $\eta$, $f$ and $f_0$ are the stream functions occurring in the velocity equation, $\sigma$ is the Prandtl/Schmidt number and $B$ is the value of $b$ in the main-stream.

From equation (8) it may readily be shown that the distribution of $b$ is given by:

$$\frac{b}{B} = \frac{\int_0^\eta \exp - \{\sigma \int_0^\eta f \, d\eta\} \, d\eta}{\int_0^\infty \exp - \{\sigma \int_0^\infty f \, d\eta\} \, d\eta} \qquad (11)$$

and the reciprocal of the wall gradient is given by:

$$\left(\frac{B}{b'_0}\right) = \int_0^\infty \exp - \{\sigma \int_0^\eta f \, d\eta\} \, d\eta. \qquad (12)$$

The present paper is concerned with methods of evaluating the right-hand side of equation (12) for any value of $\sigma$, given solutions to equation (5) with boundary conditions (6). Since such solutions are given for fixed values of $f_0$, and in view of the relationship given in equation (10), the problem amounts to evaluating the mass transfer driving force $B$, given the fluid property $\sigma$ and the velocity, specified by $f_0$, at which mass flows through the interface. The main-stream pressure gradient, of course, also affects the distribution of $f$ with $\eta$ and thereby the value of $B$ obtained using equations (12) and (10).

The table of values of $(b'_0/B)$ to be given later apply to the case when the parameter $\beta$ occurring in equation (5) is zero.

As well as evaluating the right-hand side of equation (12) the method of calculation could also be adapted to obtaining the distribution of $b$ with $\eta$ as given in equation (11) but this is not done in the present paper.

## 3. EVALUATING THE INTEGRAL

### 3.1. *Formula for numerical integration*

It is known that the relative thicknesses of the velocity boundary layer and the $b$-boundary layer depend on the value of the Prandtl/Schmidt number $\sigma$. For the unique case when the main-stream pressure gradient is zero and $\sigma$ is unity the two boundary layers are identical. This may be seen by comparing equation (5) for $\beta = 0$ with equation (8) for $\sigma = 1$. The distribution in the boundary layer of the velocity, which is $f'$, is then the same as the distribution of the quantity $(b/B)$ since they obey the same differential equation with the same boundary conditions.

When $\sigma$ is much greater than unity, however, the $b$-boundary layer is much thinner than the velocity layer, whereas when $\sigma$ is much smaller than unity the $b$-boundary layer is much thicker than the velocity layer.

For low values of $\sigma$, therefore, an appreciable contribution to the integral in equation (12) comes from regions in the main-stream where, of course, the flow is inviscid.

The integral in equation (12) can therefore be evaluated in two parts, which will be referred to as Parts I and II respectively.

*Part I:* extends over the region of the velocity boundary layer and is evaluated by standard procedures for numerical integration; this part is important for large values of $\sigma$.

*Part II:* which extends over regions of main-stream flow, can be expressed in closed form and so is readily evaluated using standard mathematical tables; this part is important for small values of $\sigma$.

For intermediate values of $\sigma$ an appreciable contribution to the integral comes from both parts.

For main-stream flow the stream function $f$ takes on a simple form which is obtained as follows.

In the main-stream $(df/d\eta) = 1$ so that $f$ must be a linear function of $\eta$. If the displacement thickness defined in terms of the "similar" distance $\eta$ is given by:

$$\delta_1^* = \int_0^\infty \left(1 - \frac{df}{d\eta}\right) d\eta, \qquad (13)$$

on integrating this formally it is found that for large values of $\eta$ the stream function takes the form:

$$f = (\eta + f_0 - \delta_1^*). \qquad (14)$$

When the fluid density is uniform the difference in the values of the stream-function at two points in any fluid stream can be regarded as a measure of the amount of fluid which crosses a line joining the two points. Equation (14) conforms with this interpretation since $f$ has the value $f_0$ at the interface where $\eta = 0$ and the amount of fluid flowing across the line joining the interface and the point $\eta$ is proportional to $(\eta - \delta_1^*)$, since $\delta_1^*$ measures the thickness of the velocity layer (i.e. $\delta_1^*$ is the distance through which the main-stream has been *displaced* by the presence of the boundary layer).

In the integral of equation (12), therefore, let Part I be integrated over the range $0 \leqslant \eta \leqslant d$ and Part II over the range $d \leqslant \eta \leqslant \infty$. If the value of $d$ is large enough for equation (14) to be valid at that point the expression for $(B/b_0')$ may be written accurately as:

$$\left(\frac{B}{b_0'}\right) = \int_0^d \exp - \left\{ \sigma \int_0^\eta f \, d\eta \right\} d\eta + \left(\frac{\pi}{2\sigma}\right)^{1/2} \left\{ 1 \pm \operatorname{erf}\left[ \left(\frac{\sigma}{2}\right)^{1/2} f(d) \right] \right\} \exp\left\{ \frac{\sigma}{2} \left[ f(d) \right]^2 - \sigma \int_0^d f \, d\eta \right\} \tag{15}$$

where $f(d)$, the value of $f$ at $\eta = d$, is:

$$f(d) = (d + f_0 - \delta_1^*). \tag{16}$$

In equation (15) the first term on the right-hand side is Part I, and the second term Part II. In Part II the sign preceding the error function should be opposite to that of $f(d)$. In practice, however, $f(d)$ is almost always positive so that this sign is generally negative; the alternative sign is included so that the formula may retain its general form.

The values of $(b_0'/B)$ to be given later were obtained using equation (15). Use of this formula is reasonably straightforward and accurate; methods of doing so will now be described.

### 3.2. *Integration using a desk calculator*

To apply this method solutions to the velocity equation are needed in the form of values of the stream function $f$ at regular intervals in $\eta$. Many such solutions are given in the literature.

Starting from such a table the first step is to construct a table of the function $\int_0^\eta f \, d\eta$ at regular intervals in $\eta$ up to a value where equation (14) holds accurately; this point is readily located from the values of $f$ and $f_0$. This table has to be obtained only once, of course, as it can then be used to evaluate $(b_0'/B)$ for any value of $\sigma$.

It should be noted that the displacement thickness $\delta_1^*$, although by definition a quantity which is obtained by integrating throughout the velocity boundary layer, can be obtained from the value of $f$ at large $\eta$ without the need for such integration. Using this and the known values of $\beta$, $f_0$ and $f_0''$, the other functions associated with the velocity layer, which were discussed in Papers 1 and 2, can also be evaluated. This procedure has proved to be particularly useful for calculations with a computer; the formulae which are used are given in the Appendix.

The table of $\int_0^\eta f \, d\eta$ is obtained in the following way. The stream function $f$ is first expanded as a Maclaurin series in terms of its gradients at $\eta = 0$, the higher derivations of $f$ needed for this being obtained by successive differentiation of equation (5). For small values of $\eta$ values of $\int_0^\eta f \, d\eta$ can be obtained directly from this series although the accuracy decreases with increasing $\eta$. At some value of $\eta$, where it is judged that this expansion is becoming too inaccurate, the method is changed to a step by step application of Simpson's rule using the tabulated values of $f$. More accurate integration formulae than Simpson's rule can obviously be used if rapid means of computation are available.

Having obtained values of $\int_0^\eta f \, d\eta$ all the functions required for evaluating Part II of equation (15) are known.

Part I is evaluated numerically by again applying Simpson's rule in an obvious manner using the table of values of $\int_0^\eta f \, d\eta$. When mass transfer is zero or inwards and $\sigma$ is large the $b$-boundary layer is confined to low values of $\eta$. To obtain high accuracy therefore a small interval in $\eta$ must be used.

On the other hand, where mass transfer is outwards and $\sigma$ is large the integrand occurring in Part I starts from the value unity at the wall, increases to a maximum within the boundary layer and then decreases to a very low value as the main-stream is approached. Since the gradient of $\int_0^\eta f\,d\eta$ with respect to the co-ordinate $\eta$ is $f$ it may readily be shown that this maximum occurs at the point where $f$ is zero. Most of the contribution to the integral therefore comes from regions within the boundary layer but not near the wall. It would therefore be surprising if an expansion in terms of wall gradients gave high accuracy, although expansion about the point where $f$ is zero should be possible. This will not be considered further in the present paper.

### 3.3. *Integration by a computer*

The method for evaluating $(b_0'/B)$ by a computer differed in many respects from that just described but since the procedure adopted was fairly conventional only a brief outline will be given here.

The computer programme was designed to be as general as possible so that values of $(b_0'/B)$ could be obtained for any suitable values of the three parameters $\beta$, $f_0$ and $\sigma$. The only data required from solutions to the velocity equation were corresponding values of $f_0$, $f_0''$, and $\delta_1^*$. In the calculations for the case $\beta = 0$ these quantities were taken from the literature.

For most of the calculations an interval of 0·1 in the independent variable $\eta$ was used. At high suction rates, namely for large positive values of $f_0$, the velocity layer is very thin so the interval was reduced to 0·05 for those cases.

At each step in the integration procedure the velocity equation was solved using a fourth order Runge–Kutta process and the functions $f$, $f'$, $f''$, $\int_0^\eta f\,d\eta$ and $\int_0^\eta \exp -\{\sigma_i \int_0^\eta f\,d\eta\}\,d\eta$ were evaluated. For any particular solution to the velocity equation the computer dealt with about 40 values of $\sigma$ simultaneously and $\sigma_i$, with $i = 1, 2, 3 \ldots$, signifies this.

At every sixth step Part I and Part II of $(B/b_0')$ occurring in equation (15) were evaluated and summed. This sum is an estimate of $(B/b_0')$ up to that value of $\eta$. This was compared with the value obtained six steps previously. If the modulus of the difference between these two values was less than or equal to $5 \times 10^{-5}$ of the last value obtained the integration was stopped for that value of $\sigma$. The programme continued until $(B/b_0')$ had been evaluated for all values of $\sigma$.

It is now realized, of course, that such a testing procedure was not necessary since the integration could have continued up to the value $\eta = d$ and stopped there for all values of $\sigma$. The method just described was used because the form of Part II of the integral in equation (15) was not known when the programme was first devised.

### 3.4. *Asymptotic formula for high $\sigma$ when $f_0$ is large and positive*

In Paper 3a, Evans [4], which was concerned with the case when no mass flows through the interface, asymptotic series in inverse powers of $\sigma$ were given from which the wall gradient $(b_0'/B)$ could be evaluated accurately for any value of $\sigma$ greater than 0·5. An expansion of the same type will now be derived for large positive $f_0$. It has not been possible to do so for negative $f_0$ because of the behaviour of the integrand; this was discussed at the end of section 3.2.

The expansion is obtained in terms of the wall gradients of the stream function $f$. The wall gradient $(b_0'/B)$ relating to the $b$-boundary layer is regarded as a single entity as expressed in equation (12) and not in two parts as given in equation (15). The accuracy of the expansion improves when each of the parameters $\sigma$ and $f_0$ increases.

When $f_0$ is large and positive and $\sigma$ is large, the driving force for mass transfer, denoted by $B$, approaches $-1$ from above. No physical meaning can be attached to values of $B$ beyond $-1$. If then the quantity $(1 + B)$ can be evaluated to a certain accuracy for known values of $\beta$, $f_0$ and $f_0''$, the functions $B$ and $(b_0'/B)$ which are calculated from these, will be known to an even higher percentage accuracy. The expansion obtained below expresses $(1 + B)$ as a series in inverse powers of $\sigma$.

Expanding the stream function $f$ in terms of wall gradients the integral $\int_0^\eta f\,d\eta$ takes the form:

$$\int_0^\eta f\,d\eta = f_0\,\eta + \frac{f_0''\eta^3}{3!} + \frac{f_0'''\eta^4}{4!} + \frac{f_0^{\text{IV}}\eta^5}{5!} + \ldots \quad (17)$$

This is now substituted into equation (12) and the integration variable changed from $\eta$ to $\varphi$ where:

$$\varphi = \sigma f_0 \eta. \tag{18}$$

Using the relationship:

$$B = -\frac{\sigma f_0}{(b_0'/B)} \tag{19}$$

equation (12) yields the expression:

$$-B = \int_0^\infty e^{-\varphi} \exp - \left\{ \sum_{q=3}^{\infty} A_q \varphi^q \right\} d\varphi \tag{20}$$

in which the coefficients $A_q$ are given by the general formula:

$$A_q = \frac{\sigma f_0^{(q-1)}}{q!} \frac{1}{(\sigma f_0)^q} \tag{21}$$

where $f_0^{(q-1)}$ denotes the $(q-1)$th derivative of $f$ with respect to $\eta$ when evaluated at $\eta = 0$.

By expanding the second exponential in the integral of equation (20) in powers of $\varphi$ and using the relationship:

$$\int_0^\infty e^{-\varphi} \varphi^p \, d\varphi = p! \tag{22}$$

which holds for any integral value of $p$, an expansion is obtained for $(1 + B)$. On collecting terms in the same inverse powers of $\sigma$ this expansion becomes:

$$(1 + B) = \frac{a_2}{\sigma^2} + \frac{a_3}{\sigma^3} + \frac{a_4}{\sigma^4} + \ldots \tag{23}$$

Using the abbreviations:

$$c = \frac{f_0''}{f_0^3} \tag{24}$$

$$e = \frac{\beta}{f_0} \tag{25}$$

the first eight coefficients in this expansion are:

$$a_2 = c \tag{26}$$

*Table 1. Values of the wall gradient $(b_0'/B$*

| $\sigma$ | $f_0$ = 10·0<br>$f_0''$ = 10·0492<br>$\delta_1^*$ = 0·098797 | $3\sqrt{2}$<br>4·35164<br>0·222053 | $2{\cdot}5\sqrt{2}$<br>3·66267<br>0·260703 | $1{\cdot}5\sqrt{2}$<br>2·30831<br>0·393434 | $0{\cdot}75\sqrt{2}$<br>1·33703<br>0·611810 | $0{\cdot}5\sqrt{2}$<br>1·03077<br>0·740107 | $0{\cdot}25\sqrt{2}$<br>0·739383<br>0·925949 |
|---|---|---|---|---|---|---|---|
| 0·0001 | 0·861969(−2) | 0·823657(−2) | 0·818854(−2) | 0·808927(−2) | 0·800763(−2) | 0·797698(−2) | 0·794290(− |
| 0·0002 | 0·125740(−1) | 0·118008(−1) | 0·117042(−1) | 0·115050(−1) | 0·113418(−1) | 0·112807(−1) | 0·112123(− |
| 0·0005 | 0·211078(−1) | 0·191406(−1) | 0·188972(−1) | 0·183959(−1) | 0·179864(−1) | 0·178340(−1) | 0·176651(− |
| 0·001 | 0·318544(−1) | 0·278467(−1) | 0·273542(−1) | 0·263446(−1) | 0·255260(−1) | 0·252220(−1) | 0·248844(− |
| 0·002 | 0·491721(−1) | 0·409560(−1) | 0·399591(−1) | 0·379212(−1) | 0·362786(−1) | 0·356717(−1) | 0·350027(− |
| 0·005 | 0·912630(−1) | 0·698160(−1) | 0·672610(−1) | 0·620743(−1) | 0·579451(−1) | 0·564373(−1) | 0·547876(− |
| 0·01 | 0·151737 | 0·107073 | 0·101803 | 0·912293(−1) | 0·829225(−1) | 0·799230(−1) | 0·766709(− |
| 0·02 | 0·262281 | 0·168678 | 0·157771 | 0·136087 | 0·119334 | 0·113377 | 0·106992 |
| 0·04 | 0·471320 | 0·274613 | 0·251880 | 0·207115 | 0·173211 | 0·161387 | 0·148901 |
| 0·06 | 0·675149 | 0·371591 | 0·336563 | 0·267946 | 0·216628 | 0·198965 | 0·180508 |
| 0·08 | 0·876936 | 0·464359 | 0·416717 | 0·323693 | 0·254745 | 0·231256 | 0·206911 |
| 0·1 | 0·107768(1) | 0·554669 | 0·494176 | 0·376309 | 0·289548 | 0·260232 | 0·230055 |
| 0·2 | 0·207609(1) | 0·989656 | 0·862717 | 0·616249 | 0·438226 | 0·379652 | 0·320693 |
| 0·3 | 0·307231(1) | 0·141399(1) | 0·121859(1) | 0·838959 | 0·566879 | 0·478686 | 0·391302 |
| 0·4 | 0·406815(1) | 0·183489(1) | 0·156999(1) | 0·105450(1) | 0·686264 | 0·568038 | 0·452110 |
| 0·5 | 0·506425(1) | 0·225452(1) | 0·191953(1) | 0·126639(1) | 0·800351 | 0·651679 | 0·507066 |
| 0·6 | 0·606060(1) | 0·267374(1) | 0·226830(1) | 0·147627(1) | 0·911078 | 0·731569 | 0·558066 |
| 0·7 | 0·705730(1) | 0·309293(1) | 0·261679(1) | 0·168500(1) | 0·101954(1) | 0·808828 | 0·606209 |
| 0·8 | 0·805430(1) | 0·351226(1) | 0·296527(1) | 0·189307(1) | 0·112641(1) | 0·884158 | 0·652184 |
| 0·9 | 0·905153(1) | 0·393181(1) | 0·331387(1) | 0·210078(1) | 0·123214(1) | 0·958030 | 0·696464 |
| 1·0 | 0·100490(2) | 0·435162(1) | 0·366265(1) | 0·230831(1) | 0·133703(1) | 0·103077(1) | 0·739382 |
| 1·2 | 0·120443(2) | 0·519199(1) | 0·436087(1) | 0·272330(1) | 0·154509(1) | 0·117377(1) | 0·822041 |
| 1·4 | 0·140403(2) | 0·603334(1) | 0·506006(1) | 0·313857(1) | 0·175173(1) | 0·131444(1) | 0·901516 |
| 1·6 | 0·160366(2) | 0·687555(1) | 0·575989(1) | 0·355433(1) | 0·195760(1) | 0·145357(1) | 0·978645 |
| 1·8 | 0·180331(2) | 0·771850(1) | 0·646049(1) | 0·397066(1) | 0·216308(1) | 0·159165(1) | 0·105399(1) |
| 2·0 | 0·200308(2) | 0·856207(1) | 0·716168(1) | 0·438757(1) | 0·236841(1) | 0·172901(1) | 0·112795(1) |
| 2·5 | 0·250280(2) | 0·106732(2) | 0·891776(1) | 0·543219(1) | 0·288193(1) | 0·207067(1) | 0·130845(1) |
| 3·0 | 0·300249(2) | 0·127867(2) | 0·106766(2) | 0·647966(1) | 0·339649(1) | 0·241127(1) | 0·148475(1) |
| 3·5 | 0·350222(2) | 0·149018(2) | 0·124372(2) | 0·752938(1) | 0·391244(1) | 0·275180(1) | 0·165841(1) |
| 4·0 | 0·400201(2) | 0·170180(2) | 0·141992(2) | 0·858088(1) | 0·442978(1) | 0·309271(1) | 0·183038(1) |
| 4·5 | 0·450182(2) | 0·191352(2) | 0·159620(2) | 0·963379(1) | 0·494842(1) | 0·343423(1) | 0·200123(1) |
| 5·0 | 0·500167(2) | 0·212530(2) | 0·177256(2) | 0·106879(2) | 0·546824(1) | 0·377643(1) | 0·217136(1) |
| 6·0 | 0·600143(2) | 0·254900(2) | 0·212545(2) | 0·127985(2) | 0·651087(1) | 0·446295(1) | 0·251044(1) |
| 7·0 | 0·700125(2) | 0·297285(2) | 0·247850(2) | 0·149115(2) | 0·755676(1) | 0·515211(1) | 0·284894(1) |
| 8·0 | 0·800112(2) | 0·339678(2) | 0·283166(2) | 0·170262(2) | 0·860513(1) | 0·584358(1) | 0·318751(1) |
| 9·0 | 0·900100(2) | 0·382078(2) | 0·318489(2) | 0·191422(2) | 0·965544(1) | 0·653702(1) | 0·352651(1) |
| 10 | 0·100009(3) | 0·424483(2) | 0·353818(2) | 0·212591(2) | 0·107074(2) | 0·723212(1) | 0·386611(1) |
| 15 | 0·150006(3) | 0·636547(2) | 0·530513(2) | 0·318516(2) | 0·159816(2) | 0·107249(2) | 0·557442(1) |
| 20 | 0·200005(3) | 0·848643(2) | 0·707246(2) | 0·424507(2) | 0·212686(2) | 0·142340(2) | 0·729759(1) |

The values given in the table must be multiplied by the powers of ten given in brackets.
Values within the region marked off by broken lines may be inaccurate (see text).

$$a_3 = -c - e \tag{27}$$

$$a_4 = c + e - 10c^2 \tag{28}$$

$$a_5 = -c - e + (34 + 2\beta)c^2 + 35ce \tag{29}$$

$$a_6 = c + e - (86 + 8\beta)c^2 - (122 + 6\beta)ce - 35e^2 + 280c^3 \tag{30}$$

$$a_7 = -c - e + (194 + 22\beta)c^2 + (317 + 26\beta)ce + (122 + 6\beta)e^2 - (2016 + 168\beta)c^3 - 2100c^2e \tag{31}$$

$$a_8 = c + e - (414 + 52\beta)c^2 - (732 + 76\beta)ce - (317 + 26\beta)e^2 + (9596 + 1348\beta + 20\beta^2)c^3 + (15\,480 + 1140\beta)c^2e + 5775ce^2 - 15\,400c^4 \tag{32}$$

$$a_9 = -c - e + (858 + 114\beta)c^2 + (1591 + 188\beta)ce + (732 + 76\beta)e^2 - (38\,100 + 6892\beta + 176\beta^2)c^3 - (75\,528 + 9608\beta + 132\beta^2)c^2e - (43\,065 + 2970\beta)ce^2 + (190\,960 + 18\,480\beta)c^4 - 5775c^3 + 200\,200c^3e. \tag{33}$$

For very large values of $f_0$ it is possible to neglect terms which contain $f^v$ and higher powers in the denominator. The remaining terms are then readily summed giving for $(1 + B)$ the much simpler expression:

$$(1 + B) = \frac{c}{\sigma(1 + \sigma)} - \frac{e}{\sigma^2(1 + \sigma)}. \tag{34}$$

ı *function of the parameters $f_0$ and $\sigma$ for $\beta = 0$.*

| 0·1√2<br>0·574341<br>1·08279 | 0·0<br>0·469600<br>1·21672 | −0·1√2<br>0·370008<br>1·38498 | −0·25√2<br>0·232624<br>1·73941 | −0·375√2<br>0·132388<br>2·21109 | −0·5√2<br>0·050229<br>3·10476 | −0·6√2<br>0·004748<br>5·55229 | σ |
|---|---|---|---|---|---|---|---|
| 0·791959(−2) | 0·790224(−2) | 0·788275(−2) | 0·784747(−2) | 0·780735(−2) | 0·774178(−2) | 0·758806(−2) | 0·0001 |
| 0·111662(−1) | 0·111316(−1) | 0·110929(−1) | 0·110229(−1) | 0·109436(−1) | 0·108150(−1) | 0·105155(−1) | 0·0002 |
| 0·175503(−1) | 0·174650(−1) | 0·173697(−1) | 0·171975(−1) | 0·170036(−1) | 0·166922(−1) | 0·159824(−1) | 0·0005 |
| 0·246567(−1) | 0·244880(−1) | 0·243001(−1) | 0·239619(−1) | 0·235838(−1) | 0·229820(−1) | 0·216426(−1) | 0·001 |
| 0·345524(−1) | 0·342204(−1) | 0·338521(−1) | 0·331919(−1) | 0·324612(−1) | 0·313151(−1) | 0·288372(−1) | 0·002 |
| 0·536860(−1) | 0·528805(−1) | 0·519918(−1) | 0·504188(−1) | 0·487064(−1) | 0·460885(−1) | 0·407257(−1) | 0·005 |
| 0·745198(−1) | 0·729570(−1) | 0·712471(−1) | 0·682540(−1) | 0·650540(−1) | 0·602897(−1) | 0·510360(−1) | 0·01 |
| 0·102818 | 0·998168(−1) | 0·965627(−1) | 0·909560(−1) | 0·850970(−1) | 0·766603(−1) | 0·613105(−1) | 0·02 |
| 0·140867 | 0·135161 | 0·129051 | 0·118728 | 0·108251 | 0·937787(−1) | 0·694246(−1) | 0·04 |
| 0·168766 | 0·160497 | 0·151720 | 0·137095 | 0·122549 | 0·103022 | 0·718451(−1) | 0·06 |
| 0·191557 | 0·180821 | 0·169505 | 0·150852 | 0·132593 | 0·108630 | 0·719143(−1) | 0·08 |
| 0·211164 | 0·198031 | 0·184270 | 0·161791 | 0·140084 | 0·112130 | 0·707666(−1) | 0·1 |
| 0·284811 | 0·260351 | 0·235249 | 0·195593 | 0·159157 | 0·115446 | 0·588487(−1) | 0·2 |
| 0·338878 | 0·303712 | 0·268160 | 0·213326 | 0·164775 | 0·109634 | 0·456862(−1) | 0·3 |
| 0·383480 | 0·337956 | 0·292480 | 0·223728 | 0·164761 | 0·101000 | 0·344045(−1) | 0·4 |
| 0·422345 | 0·366675 | 0·311650 | 0·229906 | 0·161816 | 0·915567(−1) | 0·254141(−1) | 0·5 |
| 0·457308 | 0·391675 | 0·327344 | 0·233353 | 0·157202 | 0·821497(−1) | 0·185098(−1) | 0·6 |
| 0·489421 | 0·413909 | 0·340517 | 0·234908 | 0·151608 | 0·731757(−1) | 0·133326(−1) | 0·7 |
| 0·519353 | 0·434045 | 0·351770 | 0·235091 | 0·145447 | 0·648219(−1) | 0·951733(−2) | 0·8 |
| 0·547553 | 0·452504 | 0·361506 | 0·234249 | 0·138983 | 0·571669(−1) | 0·674334(−2) | 0·9 |
| 0·574340 | 0·469599 | 0·370010 | 0·232623 | 0·132390 | 0·502297(−1) | 0·474804(−2) | 1·0 |
| 0·624569 | 0·500533 | 0·384118 | 0·227678 | 0·119265 | 0·384281(−1) | 0·231853(−2) | 1·2 |
| 0·671350 | 0·528095 | 0·395254 | 0·221211 | 0·106661 | 0·291124(−1) | 0·111454(−2) | 1·4 |
| 0·715505 | 0·553073 | 0·404147 | 0·213798 | 0·948634(−1) | 0·218818(−1) | 0·529480(−3) | 1·6 |
| 0·757590 | 0·575994 | 0·411282 | 0·205813 | 0·839989(−1) | 0·163402(−1) | 0·249258(−3) | 1·8 |
| 0·797998 | 0·597234 | 0·417006 | 0·197509 | 0·741075(−1) | 0·121353(−1) | 0·119008(−3) | 2·0 |
| 0·893408 | 0·644638 | 0·426652 | 0·176388 | 0·534947(−1) | 0·565916(−2) | 0·174046(−4) | 2·5 |
| 0·982946 | 0·685961 | 0·431503 | 0·155889 | 0·380732(−1) | 0·258539(−2) | 0·242495(−5) | 3·0 |
| 0·106831(1) | 0·722840 | 0·433034 | 0·136746 | 0·268051(−1) | 0·116374(−2) | 0·349591(−6) | 3·5 |
| 0·115055(1) | 0·756303 | 0·432160 | 0·119268 | 0·187095(−1) | 0·518048(−3) | 0·472703(−7) | 4·0 |
| 0·123039(1) | 0·787046 | 0·429495 | 0·103544 | 0·129671(−1) | 0·228659(−3) | 0·650696(−8) | 4·5 |
| 0·130831(1) | 0·815561 | 0·425468 | 0·895475(−1) | 0·893467(−2) | 0·100252(−3) | 0·889758(−9) | 5·0 |
| 0·145980(1) | 0·867277 | 0·414497 | 0·663457(−1) | 0·418286(−2) | 0·189793(−4) | 0·163868(−10) | 6·0 |
| 0·160700(1) | 0·913470 | 0·400947 | 0·486592(−1) | 0·193029(−2) | 0·353953(−5) | 0·297344(−12) | 7·0 |
| 0·175116(1) | 0·955414 | 0·385849 | 0·353998(−1) | 0·881242(−3) | 0·652947(−6) | 0·533743(−14) | 8·0 |
| 0·189307(1) | 0·993968 | 0·369870 | 0·255831(−1) | 0·398992(−3) | 0·119460(−6) | 0·950259(−16) | 9·0 |
| 0·203330(1) | 0·102974(1) | 0·353460 | 0·183865(−1) | 0·179467(−3) | 0·217144(−7) | 0·168095(−17) | 10 |
| 0·272007(1) | 0·117962(1) | 0·273194 | 0·332596(−2) | 0·310040(−5) | 0·404659(−11) | 0·273508(−26) | 15 |
| 0·339674(1) | 0·129881(1) | 0·204611 | 0·566654(−3) | 0·503795(−7) | 0·709825(−15) | 0·418995(−35) | 20 |

## 4. VALUES OF THE WALL GRADIENT $(b_0'/B)$ WHEN THE MAIN-STREAM PRESSURE GRADIENT IS ZERO

### 4.1. *General discussion of Table* 1

Values of the wall gradient $(b_0'/B)$ as a function of the fluid property group $\sigma$ and the mass flow parameter $f_0$ for the case $\beta = 0$ are given in Table 1. The solutions to the velocity equation from which these were calculated, namely corresponding values of $f_0$, $f_0''$, and $\delta_1^*$, were taken from Emmons and Leigh [6] except for the case $f_0 = 10$ which was taken from Watson [7]. The values of these parameters are given at the head of the appropriate column in Table 1.

Most of the values of $(b_0'/B)$ were calculated by a computer using the method described in section 3.3. When both $\sigma$ and $f_0$ were large, however, the values given by the computer were inaccurate because the interval in $\eta$ was too great. For the bottom left hand corner of Table 1, therefore, the values were calculated from the series expansion given in section 3.4.

For various other reasons, such as error in supplying the input data, the computer also gave incorrect values at isolated positions in other parts of the table. These values were then recalculated by the method described in section 3.2 using values of $\int_0^\eta f\,d\eta$ supplied by the computer.

### 4.2. *Accuracy of Table* 1

The method of computation described in section 3.3 was designed to give values of the wall gradient $(b_0'/B)$ with an error less than $5 \times 10^{-5}$ of its own value. Generally, therefore, the error in the fifth significant digit should be small but considerable error could occur in the sixth significant digit.

The accuracy of the results may be judged by comparing the value of $(b_0'/B)$ for $\sigma = 1{\cdot}0$ with the corresponding value of $f_0''$ given at the top of the table. They should be equal but generally differ by a few units in the sixth significant digit. The sixth place has therefore been retained in Table 1 but it should be emphasized that no great reliance can be placed on it. This applies particularly to negative values of $f_0$ since even the quantities $f_0''$ and $\delta_1^*$ are not known to high accuracy there.

During the present work it was found, when working with a computer and using accurate integration formulae, that very high accuracy in wall gradients is often desirable. This was also found by Eckert *et al.* [8] who, when studying the velocity equation, required the wall gradient $f_0''$ to ten significant digits in order to obtain solutions for the case $\beta = 1{\cdot}0, f_0 = -3{\cdot}0$. Knowledge of the value of the sixth significant digit in $(b_0'/B)$, although not exact, may therefore prove useful in applying the values to obtain other functions of the $b$-boundary layer.

Values for high $\sigma$ in the bottom left-hand corner of the table were calculated by the formula of section 3.4. For lower values of $\sigma$, particularly for $\sigma$ approaching unity, the values obtained by the computer were accurate. There was, however, an intermediate zone where both methods were inaccurate to some extent. By differencing the values and, where necessary, adjusting them so as to give smooth, regular differences, the error in the values given is believed to be confined to the sixth significant digit even here. This does not apply to the values in the first column because, when plotting $\Delta_2/\Delta_4$ as a function of $(v_0\Delta_2/K)$ (see Fig. 3), a few of the values obtained from this column did not form a continuous curve with points taken from the other columns. It is therefore suspected that some of the values in this column are in error in the fifth significant digit.

In the part of the table where mass flow is outwards ($f_0$ negative) and $\sigma \geqslant 2$ there may be some inaccuracy for the following reason. Here the integrand in equation (12) starts from unity at the wall, increases to a maximum and then diminishes to a low value in the main-stream. For large $\sigma$ this curve reaches a very high value and has steep gradients. In order to obtain good accuracy with such a curve a very small interval in $\eta$ should be used, whereas the values quoted were obtained with an interval of $0{\cdot}1$. No estimate has been made of the probable error due to this but it would be surprising if the values of $(b_0'/B)$ were inaccurate by more than a few per cent of their own values.

### 4.3. *Comparison with earlier values*

Of all the two-dimensional, laminar boundary layer flows considered in the literature the case when both the main-stream pressure gradient and the rate of mass transfer through the interface are zero has received the greatest attention; this is the case $f_0 = 0$ in Table 1.

In order to compare the present results for this case with exact values given in the literature Table 2 has been drawn up. This table contains some values for low $\sigma$ not included in Table 1; these were calculated by the method described in section 3.2.

*Table 2. Comparison with published values for* $\beta = 0, f_0 = 0$

(a) Present values (b) Published values

| $\sigma$ | $(b_0/B)$ (a) | $(b_0/B)$ (b) | Reference |
|---|---|---|---|
| 0·003 | 0·0415366 | 0·041537 | [10] |
| 0·006 | 0·057595 | 0·057593 | [10] |
| 0·01 | 0·072957 | 0·072959 | [10] |
| 0·03 | 0·119346 | 0·11935 | [10] |
| 0·1 | 0·198031 | 0·1953 | [9] |
| 0·5 | 0·366675 | 0·3664 | [9] |
| 0·6 | 0·391675 | 0·3915<br>0·3917 | [9]<br>[11] |
| 0·7 | 0·413909 | 0·4139<br>0·4137 | [9]<br>[10] |
| 0·8 | 0·434045 | 0·4340<br>0·4342 | [9]<br>[11] |
| 0·9 | 0·452504 | 0·4525 | [9, 11] |
| 1 | 0·469599 | 0·46960<br>0·4695 | [6]<br>[9, 11] |
| 2 | 0·597234 | 0·5971 | [11] |
| 7 | 0·913470 | 0·9135 | [9] |
| 10 | 1·02974 | 1·0298 | [9] |
| 15 | 1·17962 | 1·1796 | [9] |

The numbers in the last column give the publication in the reference list.

From this table it may be seen that values in the literature sometimes disagree with each other by a few units in the fourth significant digit. The error in the value given by Merk [9] for $\sigma = 0{\cdot}1$ is more serious, however, as it appears to be 1·4 per cent too low. The agreement between the present values and those given by Sparrow and Gregg [10] for low values of $\sigma$ is very good, which confirms the formula for Part II given in equation (15) since this Part dominates for very low $\sigma$.

This case when $\beta = 0$ and $f_0 = 0$ was also considered in Paper 3a, where an asymptotic formula was given for calculating $(b_0'/B)$ for any high value of $\sigma$. It appears from the present results that the formula gives an accuracy of one unit in the fourth significant digit even when $\sigma$ is as low as 0·5.

The only extensive results giving values of $(b_0'/B)$ in the presence of mass transfer were those by Mickley *et al.* [11], which were also for the case $\beta = 0$. On the whole the present values agree with their results to the same extent as for the case of no mass transfer given in Table 2. This does not hold, however, for large $\sigma$ and high blowing rates (large negative $f_0$). The greatest difference occurs for $f_0 = -0{\cdot}5\sqrt{2}$ and $\sigma = 5{\cdot}0$, the highest blowing rate and the largest value of $\sigma$ considered by the earlier authors. Whereas they obtained for $(b_0'/B)$ the value $0{\cdot}9963 \times 10^{-4}$, the present value is $1{\cdot}0025 \times 10^{-4}$, a difference of 0·6 per cent. A possible source of error in the present results for this part of the table has been given in section 4.2.

## 5. FORMULAE FOR OBTAINING OTHER FUNCTIONS FROM $(b_0'/B)$

It was shown in Paper 3 that many other functions relating to the *b*-boundary layer can be calculated from values of $(b_0/B)$ for known values of the parameters $\beta$, $\sigma$ and $f_0$. For ease of reference these are quoted below but will not be discussed; the reader is referred to other papers in the present series for more detailed discussion of these functions.

The right-hand sides of the following equations are written in terms of "similar" functions. Some of these expressions differ from the forms used in Paper 3; this is merely so that the quantities on the left may be evaluated directly in terms of functions occurring in Table 1 without intermediate calculation. A few of these functions are identically zero for the particular case $\beta = 0$ but are included to make the list complete.

$$\frac{Nu}{Re^{1/2}} = \frac{1}{(2-\beta)^{1/2}}\left(\frac{b_0'}{B}\right) \tag{35}$$

$$\frac{\Delta_2 Re^{1/2}}{x} = \frac{(2-\beta)^{1/2}}{\sigma}\left\{\left(\frac{b_0'}{B}\right) - \sigma f_0\right\} \tag{36}$$

$$B = -\frac{\sigma f_0}{(b_0'/B)} \tag{37}$$

$$\frac{v_0 \Delta_2}{K} = f_0 \left\{ \sigma f_0 - \left( \frac{b_0'}{B} \right) \right\} \tag{38}$$

$$\frac{\Delta_2}{\Delta_4} = \frac{1}{\sigma} \left( \frac{b_0'}{B} \right) \left\{ \left( \frac{b_0'}{B} \right) - \sigma f_0 \right\} \tag{39}$$

$$\frac{\Delta_4^2}{\nu} \frac{du_G}{dx} = \frac{\beta}{(b_0'/B)^2} \tag{40}$$

$$\frac{u_G}{\nu} \frac{d\Delta_4^2}{dx} = \frac{2(1-\beta)}{(b_0'/B)^2} \tag{41}$$

$$\frac{\Delta_2^2}{\nu} \frac{du_G}{dx} = \frac{\beta}{\sigma^2} \left\{ \left( \frac{b_0'}{B} \right) - \sigma f_0 \right\}^2 \tag{42}$$

$$\frac{u_G}{\nu} \frac{d\Delta_2^2}{dx} = \frac{2(1-\beta)}{\sigma^2} \left\{ \left( \frac{b_0'}{B} \right) - \sigma f_0 \right\}^2 \tag{43}$$

$$X \equiv \left\{ \frac{\Delta_4 \delta_4}{\nu} \frac{du_G}{dx} - \frac{v_0 \Delta_4}{\nu} \right\} = \left( f_0 + \frac{\beta}{f_0''} \right) \frac{1}{(b_0'/B)} \tag{44}$$

$$Y \equiv \frac{1}{K} \left( \frac{\delta_4}{u_G} \right)^{1/2} \frac{d}{dx} \left\{ \Delta_4^3 \left( \frac{u_G}{\delta_4} \right)^{3/2} \right\} = \frac{3}{2} \frac{\sigma f_0''}{(b_0'/B)^3} \tag{45}$$

$$W \equiv \left\{ \frac{\Delta_2^{1/2} \delta_4^{3/2}}{\nu} \frac{du_G}{dx} - \frac{\Delta_2^{1/2} \delta_4^{1/2}}{\nu} v_0 \right\}$$

$$= \left( f_0 + \frac{\beta}{f_0''} \right) \left\{ \frac{(b_0/B) - \sigma f_0}{\sigma f_0''} \right\}^{1/2} \tag{46}$$

$$Z \equiv \frac{1}{K} \left( \frac{\delta_4}{u_G} \right)^{1/2} \frac{d}{dx} (u_G \Delta_2)^{3/2}$$

$$= \frac{3}{2} \frac{1}{(\sigma f_0'')^{1/2}} \left\{ \left( \frac{b_0'}{B} \right) - \sigma f_0 \right\}^{3/2}. \tag{47}$$

Note that the functions $X$, $Y$, $W$ and $Z$ form the basis of an approximate method of estimating boundary layer thicknesses which is most accurate for large values of $\sigma$. For low $\sigma$ the use of different functions would probably result in better accuracy.

## 6. THE ASYMPTOTIC BEHAVIOUR OF FUNCTIONS IN THE *b*-BOUNDARY LAYER

In this section a brief examination will be made of the asymptotic behaviour of functions relating to the *b*-boundary layer for extreme values of some of the parameters. The conclusions arrived at will then be used in section 7 to draw curves showing the relationships between some of these functions.

### 6.1. *The limiting case* $\sigma \to 0$

On examining equation (15) it may be seen that as $\sigma \to 0$ Part I tends to the value $d$; this, in similar co-ordinates, is the distance from the interface at which the flow is virtually inviscid. At the same time, Part II tends to the value $(\pi/2\sigma)^{1/2}$ since both the factor containing the error function and the exponential factor tend to unity. For very small $\sigma$ Part I is clearly negligible compared with this, so the following approximations apply:

$$\text{As } \sigma \to 0 \quad \left\{ \begin{array}{ll} \left( \dfrac{b_0'}{B} \right) = \left( \dfrac{2\sigma}{\pi} \right)^{1/2} & (48) \\[2ex] \dfrac{\Delta_2}{\Delta_4} = \dfrac{2}{\pi} = 0{\cdot}636620. & (49) \end{array} \right.$$

It is interesting to note that equation (49) applies exactly for purely inviscid flow when $\sigma = 0$.

For small values of $\sigma$ the mass transfer rate, as measured by the value of the parameter $f_0$, has only a small effect on $(b_0'/B)$. Referring to the smallest value of $\sigma$ included in Table 1, namely $\sigma = 0{\cdot}0001$, for which $(2\sigma/\pi)^{1/2} = 0{\cdot}00797885$, the extreme values of $f_0$ considered, namely $+10$ and $-0{\cdot}6\sqrt{2}$, give values of $(b_0'/B)$ of $0{\cdot}00861969$ and $0{\cdot}00758806$ respectively. For smaller values of $\sigma$ these values of $(b_0'/B)$ would be nearer to $(2\sigma/\pi)^{1/2}$ and would therefore cover a narrower range.

### 6.2. *The limiting case* $\sigma \to \infty$

The behaviour of boundary layer functions for large $\sigma$ when no mass flows through the interface may be estimated from the asymptotic formulae given in Paper 3a. This case will not, therefore, be discussed here except to state that for accelerated and slightly decelerated flows $(b_0'/B)$ is proportional to $\sigma^{1/3}$ and for flows very near to or at the separation point $(b_0'/B)$ is proportional to $\sigma^{1/4}$.

Contrary to what happens for low values of $\sigma$, the mass transfer rate has a large effect on

$(b_0'/B)$ when $\sigma$ is large. For positive values of $f_0$, for example, the driving force $B$ is very close to $-1$ as soon as $f_0$ has a non-zero value, so that the following approximation then applies:

$$\sigma \text{ large, } f_0 \text{ positive: } \left(\frac{b_0'}{B}\right) = \sigma f_0. \tag{50}$$

When $f_0$ is negative, on the other hand, $B$ tends to infinity as soon as $f_0$ is non-zero and $(b_0'/B)$ tends to zero.

Accurate numerical values of $(b_0'/B)$ for positive $f_0$ and large $\sigma$ can be obtained from the asymptotic formula given in section 3.4. Negative values of $f_0$ would require some modification of the methods already described.

### 6.3. *Intensive suction*

When considering the velocity boundary layer the case of intensive suction gives rise to the well-known asymptotic suction profile. In an analogous manner the $b$-profile can be shown to approach the asymptotic profile:

$$\left(\frac{b}{B}\right) = (1 - e^{-\sigma f_0 \eta}). \tag{51}$$

Instead of using this, however, the case of intensive suction can be examined more accurately by referring to equation (34). Since $f_0''$ is almost equal to $f_0$ for sufficiently large $f_0$, the following relationships hold:

$$(1 + B) = \frac{1}{\sigma(1+\sigma) f_0^2} \tag{52}$$

$$\left(\frac{b_0'}{B}\right) = \frac{\sigma f_0}{1 - [1/\sigma(1+\sigma) f_0^2]} \tag{53}$$

$$\frac{v_0 \Delta_2}{K} = \frac{1}{(1/\sigma f_0) - (1+\sigma)} \tag{54}$$

$$\frac{\Delta_2}{\Delta_4} = \frac{1}{(1+\sigma) - (2/\sigma f_0^2)}\,. \tag{55}$$

When $f_0$ is very large the last two relationships reduce to:

$$-\frac{v_0 \Delta_2}{K} = \frac{\Delta_2}{\Delta_4} = \frac{1}{(1+\sigma)}\,, \tag{56}$$

which may also be obtained by evaluating the convection thickness $\Delta_2$ for the asymptotic suction profiles relating to the velocity and the function $(b/B)$ respectively.

### 6.4. "*Separation*" *on a flat plate*

When the main-stream pressure gradient is zero and the blowing rate reaches the value $f_0 = -0{\cdot}875745$, the wall shear, represented in "similar" co-ordinates by $f_0''$, becomes zero. The velocity layer is then said to separate. This case has been discussed by Emmons and Leigh [6] in giving similar solutions to the velocity equation for flow over a flat plate.

For this and all higher blowing rates the following relationship holds:

$$F_2 \equiv \frac{u_G}{\nu} \frac{d\delta_2^2}{dx} = 2 \frac{v_0 \delta_2}{\nu}\,. \tag{57}$$

A physical interpretation of this is that the momentum boundary layer thickness $\delta_2$ grows with distance $x$ at a rate which is proportional to the blowing rate, since the group $(v_0\delta_2/\nu)$, which can be shown to be numerically equal to $f_0^2$, is a suitable measure of the blowing rate (see Papers 1 and 2).

An analogous situation must also apply to the $b$-boundary layer. In terms of equation (12) the integral on the right-hand side becomes infinite so that $(b_0'/B) = 0$. Since $b_0'(= -\sigma f_0)$ is still finite this means that $B$ is infinite. This clearly holds for the case $\sigma = 1{\cdot}0$ by direct analogy with the velocity equation, but it must apply whatever the value of $\sigma$.

The following relationships therefore hold for all $f_0$ beyond the value $f_0 = -0{\cdot}875745$:

$$\left(\frac{b_0'}{B}\right) = 0 \tag{58}$$

$$B = \infty \tag{59}$$

$$\Delta_4 = 0 \tag{60}$$

$$\frac{v_0 \Delta_2}{K} = \sigma f_0^2 \tag{61}$$

$$\frac{\Delta_2}{\Delta_4} = 0 \tag{62}$$

$$\frac{u_G}{K} \frac{d\Delta_2^2}{dx} = 2 \frac{v_0 \Delta_2}{K} = 2\,\sigma f_0^2. \tag{63}$$

In view of these relationships and the success obtained with the velocity equation when the mass transfer parameter $(v_0\delta_2/\nu)$ was used, it may be an advantage, in some respects, to use the parameter $(v_0\Delta_2/K)$ instead of $B\{=(v_0\Delta_4/K)\}$ for the $b$-boundary layer.

## 7. CURVES OF FUNCTIONS OF THE $b$-BOUNDARY LAYER

### 7.1. *Variation of Nusselt number with mass transfer driving force*

Equation (35) shows that the wall gradient $(b'_0/B)$ is a measure of the local Nusselt number *Nu*. Figs. 1 and 2 show how this gradient, when multiplied by an appropriate power of $\sigma$, varies with the mass transfer driving force $B$ for various values of $\sigma$. The ordinate contains the power of $\sigma$ in order to bring closer together curves for different values of $\sigma$. Fig. 1 is concerned with high values of $\sigma$ and Fig. 2 with low values.

(a) *High values of* $\sigma$. Fig. 1 is an extension to wider ranges in the values of the driving force $B$ and the Prandtl/Schmidt number $\sigma$ of Fig. 2 in Paper 3. The behaviour of the curves has already received some discussion in that paper so only a few additional remarks will be made here.

The sections shown as broken lines for high positive values of $B$ indicate that the curves have either been drawn between widely separated points or have been extrapolated by giving them

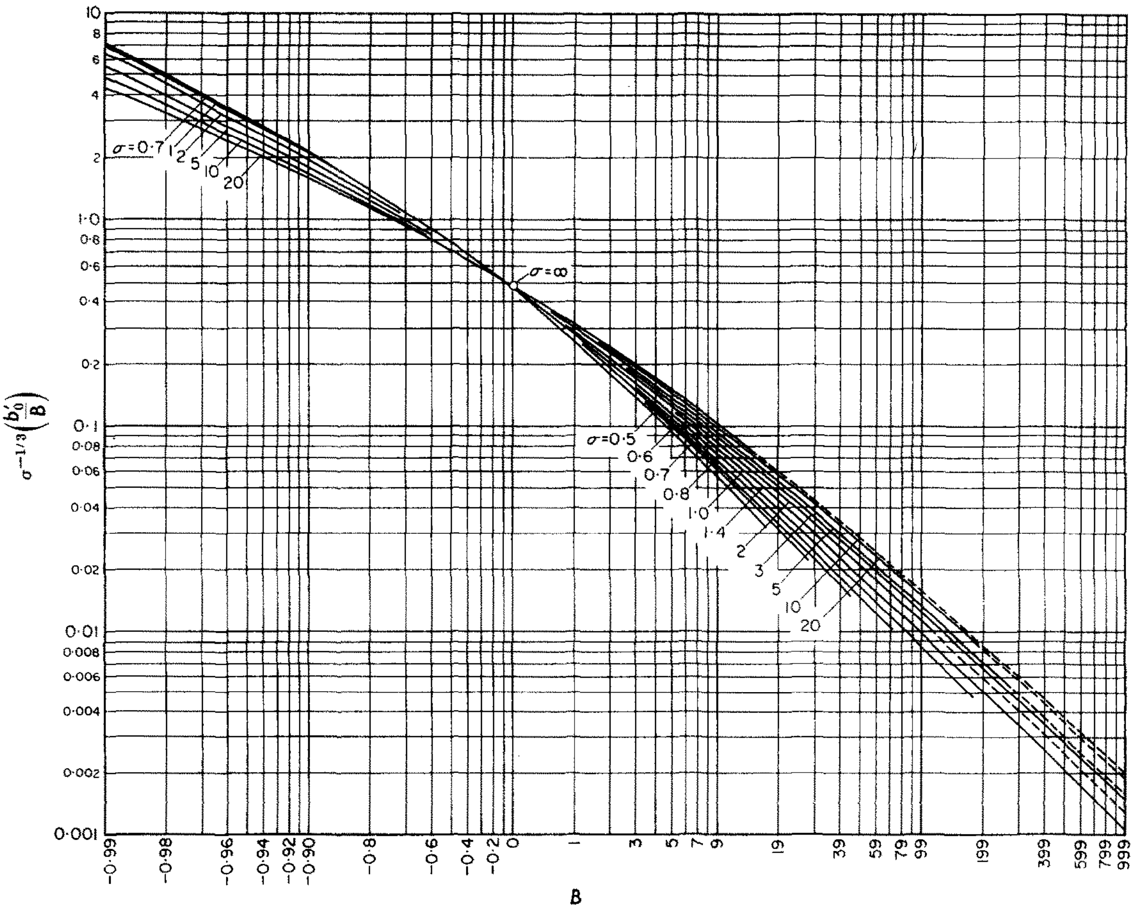


FIG. 1. Variation of the function $\sigma^{-1/3}\,(b'_0/B)$ with mass transfer driving force $B$ for high values of $\sigma$.

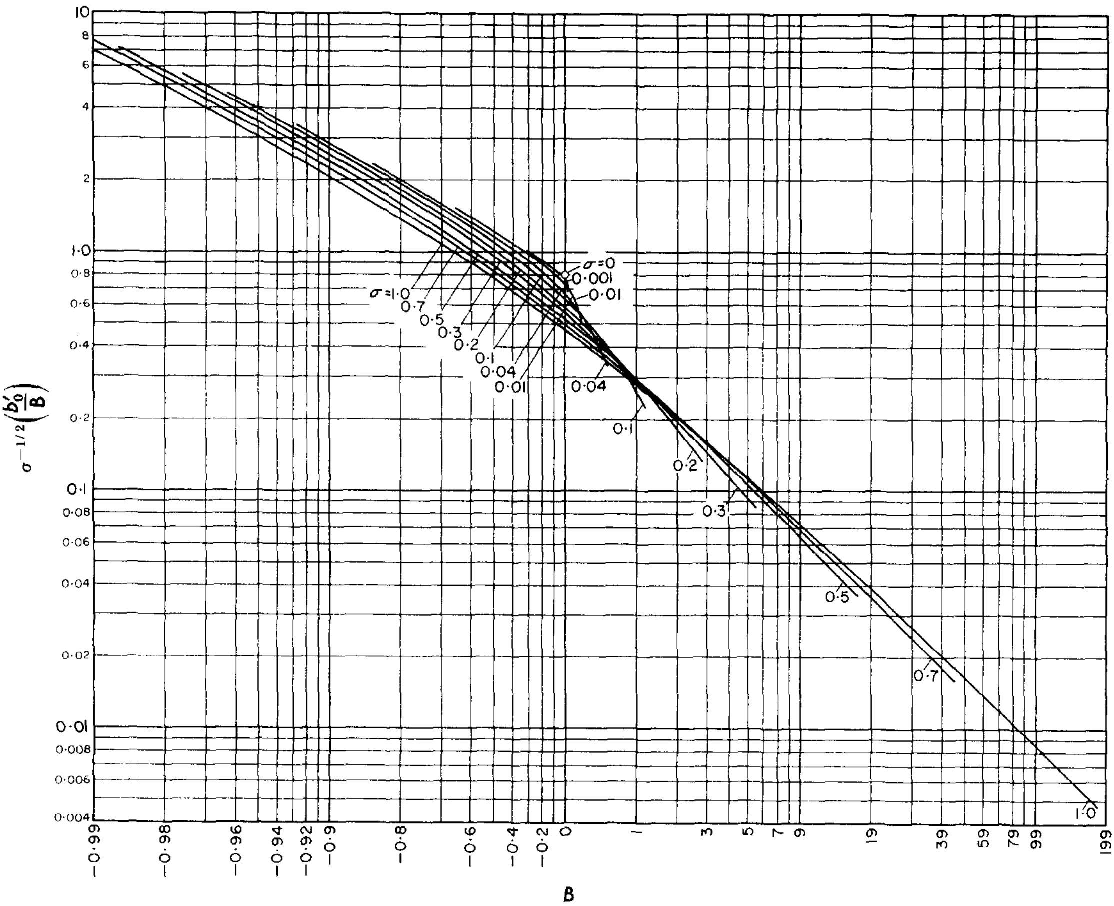


FIG. 2. Variation of the function $\sigma^{-1/2}\,(b'_0/B)$ with mass transfer driving force $B$ for low values of $\sigma$.

the same shape as neighbouring curves whose shape was known. On this scale the error brought about by this would be very small.

For negative values of $B$, some lines have been omitted for clarity. Of the values of $\sigma$ near unity, for example, only curves for $\sigma = 0{\cdot}7$ and 1·0 have been included because the curves for other values, at least down to $\sigma = 0{\cdot}5$, are virtually coincident with these.

The point indicated as $\sigma = \infty$ on the line $B = 0$ is given by:

$$\sigma^{-1/3}\left(\frac{b'_0}{B}\right) = \frac{3}{\Gamma(\frac{1}{3})}\left(\frac{f'_0}{3!}\right)^{1/3} = 0{\cdot}479017 \quad (64)$$

where $\Gamma$ denotes the gamma function. This relationship has been discussed in Paper 3a.

(b) *Low values of $\sigma$.* It was shown in section 6.1 that for small values of $\sigma$ the wall gradient $(b'_0/B)$ is proportional to $\sigma^{1/2}$. In Fig. 2, therefore the ordinate is $\sigma^{-1/2}(b'_0/B)$. The ends of the curves in this figure correspond to the extreme values of $f_0$ included in Table 1.

The most striking feature about these curves is their behaviour when $B$ is positive, i.e. when mass flows outwards through the interface. For any fixed value of $\sigma$ the ordinate quantity decreases as $B$ increases. As the value of $\sigma$ decreases, however, the curves descend more and more rapidly. From the discussion already given in section 6.4 each of these curves must tend to zero on the right when the blowing rate is high enough to cause the velocity layer to separate.

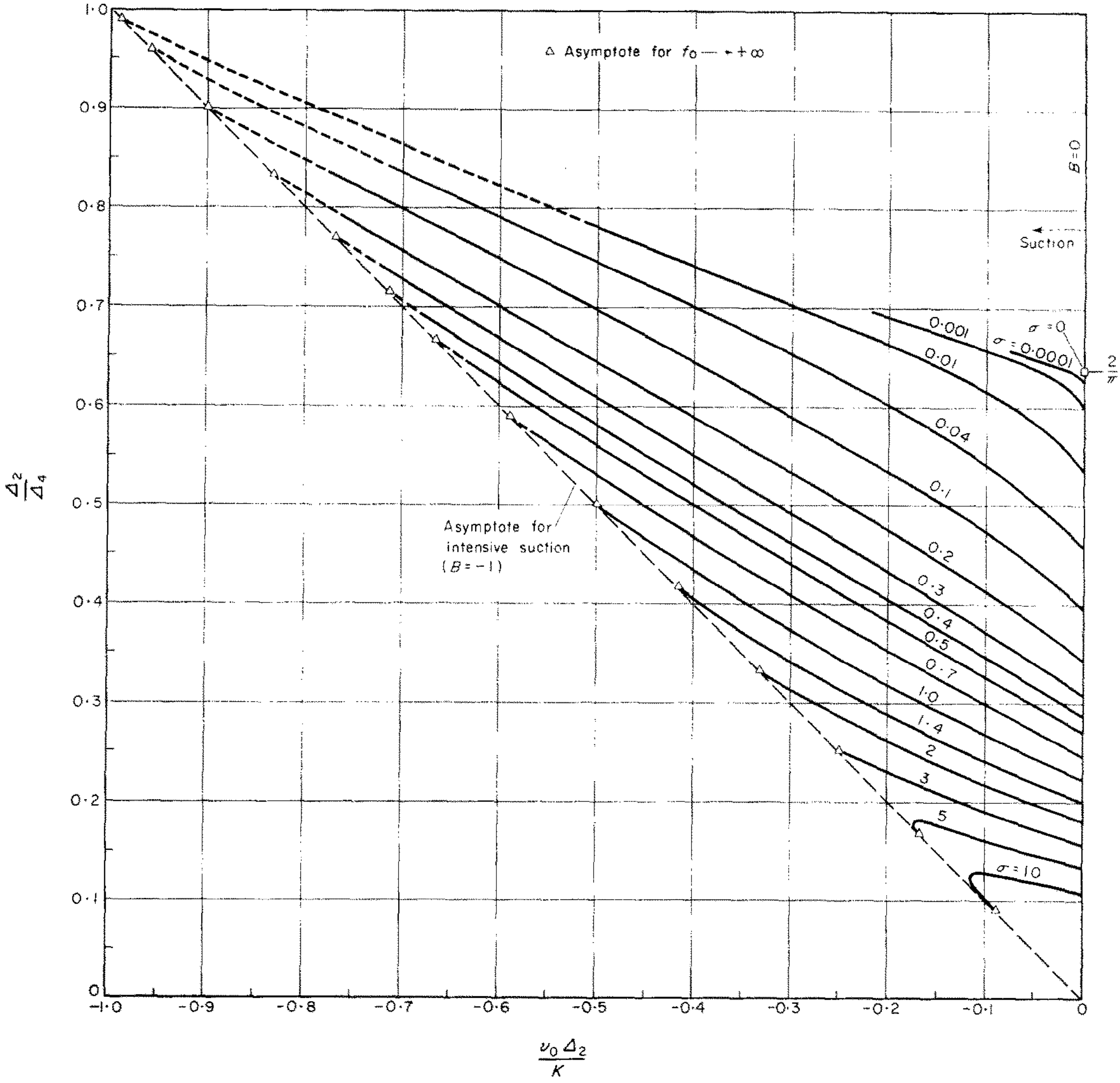


FIG. 3. Variation of the ratio of boundary layer thicknesses ($\Delta_2/\Delta_4$) with ($v_0\Delta_2/K$) when mass transfer is inwards.

The point shown as $\sigma = 0$ is given by:

$$\sigma^{-1/2}\left(\frac{b_0'}{B}\right) = \left(\frac{2}{\pi}\right)^{1/2} = 0{\cdot}797885. \quad (65)$$

7.2. *Variation of the ratio $\Delta_2/\Delta_4$ with mass transfer*

(a) *Inward mass transfer*. It has already been seen in section 6.2 that for intensive inward mass transfer the driving force $B$ approaches $-1$ and the wall gradient ($b_0'/B$) becomes very large. In order to accommodate such points on Figs. 1 and 2 the co-ordinate axes must be extended both upwards and to the left. It is not, of course, possible to include the asymptote itself since $B$ is then $-1$ and ($b_0'/B$) is infinite.

On the other hand the ratio $\Delta_2/\Delta_4$ and the mass transfer parameter $-(v_0\Delta_2/K)$ are always between 0 and 1 when the mass transfer is inwards. Fig. 3 shows the relationship between these quantities for constant values of $\sigma$. This figure is analogous to part of Fig. 2 in Paper 2 where the ratio $H_{24}(= \delta_2/\delta_4)$ was plotted as a function of the mass transfer parameter ($v_0\delta_2/\nu$)

(note that $v_s$ in the earlier paper is here denoted by $v_0$).

Since the ratio of the abscissa to the ordinate for any point on these curves gives the value of $B\{=(v_0\Delta_4/K)\}$, lines of constant $B$ are straight lines of slope $1/B$ passing through the origin.

The full lines in this figure cover the range of mass transfer included in Table 1, namely, up to an inward mass flow rate corresponding to $f_0 = +10$. The broken portions of these lines represent extrapolations to the point on the asymptote, represented by a triangle, which satisfies equation (56). Extrapolations have not been included for the two lowest values of $\sigma$ as it was difficult to judge the shapes of the curves over such a long distance. They are almost, but not exactly, linear in this region.

For any point on these curves or the extrapolations the wall gradient $(b_0'/B)$ can be calculated from the formula:

$$\left(\frac{b_0'}{B}\right) = \frac{\Delta_2}{\Delta_4} \frac{\sigma^{1/2}}{\{(v_0\Delta_2/K) + (\Delta_2/\Delta_4)\}^{1/2}} . \quad (66)$$

Clearly the accuracy of this diminishes as the asymptote is approached since $-(v_0\Delta_2/K)$ and $\Delta_2/\Delta_4$ are then almost equal.

Two points should be noted regarding the general shapes of these curves. Firstly, it was noted when discussing Fig. 2 in section 7.1 that, for the case of outward mass transfer, the ordinate descended more and more rapidly with increasing values of $B$ as the value of $\sigma$ decreased. This effect seems to be evident even on the suction side in Fig. 3. Secondly, the approach to the asymptote for intensive suction is from below when $\sigma$ is small (i.e. $\sigma \leqslant 2$), whereas when $\sigma$ is large the approach is from above.

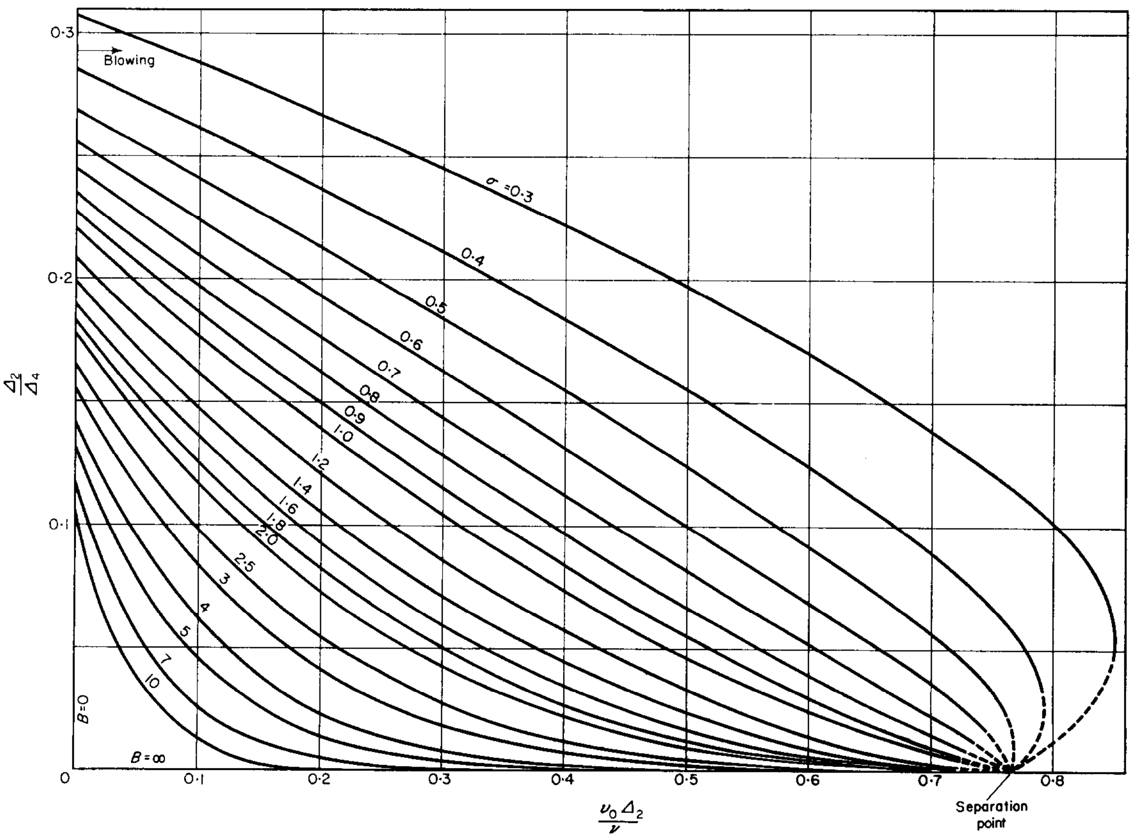


FIG. 4. Variation of the ratio $(\Delta_2/\Delta_4)$ with $(v_0\Delta_2/\nu)$ when mass transfer is outwards.

(b) *Outward mass transfer*. It was shown in section 6.4 that when the velocity boundary layer separates from the interface, $B$ is infinite and $(b'_0/B)$ is zero for all values of $\sigma$. This point also cannot be included in Figs. 1 and 2. Instead of using the mass transfer parameter $(v_0\Delta_2/K)$ along the abscissa as in Fig. 3, however, it follows from equation (61) that the parameter $(v_0\Delta_2/\nu)$ would be better since the point of separation would then be the same for all $\sigma$.

The relationship between the ratio $(\Delta_2/\Delta_4)$ and this mass transfer parameter is shown in Fig. 4. All the curves are seen to converge on the point where $(\Delta_2/\Delta_4) = 0$, $(v_0\Delta_2/\nu) = 0{\cdot}766929$.

Again the full lines represent the results contained in Table 1 and the extensions, shown as broken lines, join the point for $f_0 = -0{\cdot}6\sqrt{2}$ to the separation point. For very low values of $\sigma$ the curves form a wide arc to the right before returning to cut the abscissa.

These extrapolations to the separation point can clearly be used to extend Figs. 1 and 2 to the right. The accuracy of this would depend largely on the accuracy to which $(\Delta_2/\Delta_4)$ can be estimated since inaccuracy in the function $\{(v_0\Delta_2/K) + (\Delta_2/\Delta_4)\}^{1/2}$ occurring in equation (66) would introduce only a small error.

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## APPENDIX

The displacement and momentum boundary layer thicknesses of the velocity layer are usually evaluated by integrating certain quantities throughout the boundary layer. The stream function is, however, known in the main-stream whether solutions are taken from tables in the literature or evaluated on a computer. The formulae given below can be used for calculating functions associated with the velocity layer without the need for integration. The first two formulae are definitions of $\delta_1^*$ and $\delta_2^*$. The importance and applications of the succeeding functions have been discussed in Papers 1 and 2 of the present series.

$$\delta_1^* = \int_0^\infty \left(1 - \frac{df}{d\eta}\right) d\eta \qquad \text{(A1)}$$

$$\delta_2^* = \int_0^\infty \frac{df}{d\eta}\left(1 - \frac{df}{d\eta}\right) d\eta \qquad \text{(A2)}$$

$$\delta_1^* = \{\eta + f_0 - f\}_{\eta \text{ large}} \qquad \text{(A3)}$$

$$\delta_2^* = (f_0'' - f_0 - \beta\, \delta_1^*)/(\beta + 1) \qquad \text{(A4)}$$

$$H_{12} = \frac{\delta_1^*}{\delta_2^*} \tag{A5}$$

$$H_{24} = f_0'' \, \delta_2^* \tag{A6}$$

$$\frac{v_0 \delta_2}{\nu} = -f_0 \, \delta_2^* \tag{A7}$$

$$\frac{\delta_2^2}{\nu} \frac{\mathrm{d}u_G}{\mathrm{d}x} = \beta(\delta_2^*)^2 \tag{A8}$$

$$F_2 \equiv \frac{u_G}{\nu} \frac{\mathrm{d}\delta_2^2}{\mathrm{d}x} = 2\{1 - \beta\}(\delta_2^*)^2 \tag{A9}$$